

Two approximations are made, one essential and the other not so essential but convenient to keep the analytical treatment manageable:

1. Only one nonlinear resonance is considered at a time so that the treatment is best suited when the tune is close to one resonance only. To improve this approximation, one must go to the next order which involves a canonical transformation of dynamical variables. Analytical treatment of more than one resonance is not possible for general cases.

2. In the formalism using the action-angle variables, the Hamiltonian can have terms which are independent of the angle variables. These terms are called "phase-independent terms" or "shear terms". The tune is then a function of the oscillation amplitudes. In the lowest-order treatment, the $(4N)$ -pole components but not the $(4N+2)$ -pole components contribute to this dependence. In deriving the resonance width analytically, one ignores these terms in the Hamiltonian for the sake of simplicity. If these are retained, one needs at least three extra parameters and the analytical treatment becomes rather unwieldy. Qualitatively speaking, the phase-independent terms move the tune away from the resonance (detuning effects) as the oscillation amplitude grows so that an unstable motion can become stable at some large amplitudes (which, however, may be meaningless if the amplitude is larger than the available physical aperture). Unfortunately, they also tend to shrink the central stable area and consequently increase the resonance width. This lengthy explanation is included here to warn the readers that the resonance width given below could be an underestimate.

We start with the expression which is similar to what G. Guignard gives in his report, CERN 77-10, "SELECTION OF FORMULAE CONCERNING PROTON STORAGE RINGS", pp. 69, 70 & 76. His definition of the width seems to be regarded as the "official" definition at CERN. Consider a nonlinear sum resonance¹ characterized by three positive integers (n_x, n_y, p) . The operating point (v_x, v_y) should not be too far from this resonance line (the first of two approximations given above) so that the magnitude of the quantity

$$e \equiv n_x v_x + n_y v_y - p \quad (n_x + n_y \equiv N)$$

is not too large. This resonance is driven by the multipole field component b_{N-1} (normal field) or a_{N-1} (skew field), $N \equiv n_x + n_y$, depending on even n_x or odd n_y , respectively. The definition of b_n and a_n is

$$y = 0 \text{ (median plane)}$$

$$\text{normal field} = B_0 (1 + \sum b_n x^n),$$

$$\text{skew field} = B_0 \sum a_n x^n \quad (B_0 = \text{dipole field})$$

Now define the parameter $|d_p|$,

$$|d_p| = \left| (1/2\pi) \int \frac{ds}{\rho} (\beta_x^{n_x} \beta_y^{n_y})^{1/2} b_{N-1} \text{ (or } a_{N-1}) \cdot (N-1)! \right. \\ \left. \times \exp\{i(n_x \psi_x + n_y \psi_y - e\theta)\} \right|$$

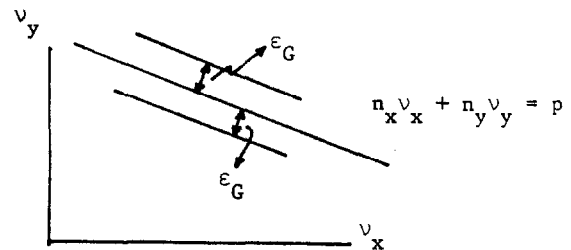
where ρ is the radius of curvature for each dipole, ψ_x and ψ_y are the linear phase advance, β_x and β_y are the linear lattice parameter, θ is the azimuthal angle of the machine and the integral is for the entire ring. The integral is usually replaced by a summation over each magnet with $(ds/\rho) = \text{bend angle}$. The resonance width of Guignard is then

$$\epsilon_G = \frac{(n_x^2 + n_y^2)^{1/2}}{2^{N-1} n_x! n_y!} E^{(N-2)/2} |d_p|$$

where the transverse beam emittance is taken to be πE in both directions. If two emittances are not identical, one must use the expression

$$G = \frac{|d_p|}{2^{N-1} n_x! n_y! (n_x^2 + n_y^2)^{1/2}} \times \\ \times \{ n_x^2 E_x (n_x-2)/2 E_y^{n_y/2} + n_y^2 E_y n_x/2 E_x^{n_x/2} E_y^{(n_y-2)/2} \}$$

However, a certain caution is required in using this expression when either n_x or n_y is unity since either E_x or E_y then appears in the denominator. Formally, one finds a very large resonance width for a very small amplitude which is of course absurd.²



Note that the "bandwidth" Δe used by Guignard is related to ϵ_G by the relation

$$\Delta e \equiv 2\epsilon_G (n_x^2 + n_y^2)^{1/2}.$$

The importance of two approximations used in the derivation of the resonance width has already been stated. When these approximations are valid, the physical meaning of ϵ_G is: If the operating point falls within the distance ϵ_G from the resonance line, a particle with the initial emittance πE becomes unstable regardless of its initial phase. The initial emittance (divided by π) of a particle in the horizontal phase space, for example, is defined with the initial values x_0 and x'_0

$$E_x \equiv \gamma_x x_0^2 + 2\alpha_x x_0 x'_0 + \beta_x x_0'^2$$

where $(\gamma_x, \alpha_x, \beta_x)$ are the linear lattice parameters.³ Note that when the distance to the resonance line is more than ϵ_G , a particle may or may not be stable depending on its initial phase. In this sense, the width ϵ_G is the most underestimated, i.e., the quantity taking the smallest value compared to other widths defined in

* Operated by the Universities Research Association, Inc., under contract with the U.S. Dept. of Energy.

different manners.

One can define the resonance width, call it ϵ_S ("S" for "safe"?), such that it is the most overestimated: If the operating point falls outside the distance ϵ_S from the resonance line, a particle with the initial emittance πE is always stable regardless of its initial phase. If the distance to the resonance line is less than ϵ_S , the motion may or may not be unstable depending on its initial phase. Obviously, for an identical situation, ϵ_S is larger than ϵ_G for any resonance. It is not possible to find an analytical expression for ϵ_S except for a few cases such as 3ν , $\nu_1 + 2\nu_2$ and $2\nu_1 + 2\nu_2$. A table is given here, therefore, showing the numerical relation between ϵ_G and ϵ_S for a number of resonances.

$$n_1 \nu_1 + n_2 \nu_2 = p, \quad n_1 < n_2 \quad \text{or} \quad n_2 = 0.$$

n_1	n_2	ϵ_G/ϵ_S	n_1	n_2	ϵ_G/ϵ_S
3	0	.500	1	5	.366
1	2	.518	7	0	.342
4	0	.414	3	4	.343
2	2	.414	2	5	.349
1	3	.433	1	6	.350
5	0	.377	8	0	.332
2	3	.380	4	4	.332
1	4	.391	3	5	.335
6	0	.355	2	6	.339
3	3	.355	1	7	.338
2	4	.362			

The width ϵ_S is termed the "most overestimated" in view of its physical meaning. This should be examined more carefully. A definite possibility of an underestimation has been mentioned in connection with phase-independent terms which shrink the central stable area in phase space. In addition, when we define the widths ϵ_G and ϵ_S , the stable motion is possible only for an arbitrarily large aperture. If the physical aperture is limited (as it always is in real machines), particles could be lost even when their motions are theoretically stable. For example, if resonance $\nu_x + 2\nu_y$ is dominant, the emittance can grow by a factor of as large as 2.9 in the horizontal direction and 4.8 in the vertical direction. This is of course for the worst initial value of the phase combination $(\psi_x + 2\psi_y)$. For other particles with different initial phases and especially with initial emittances less than the maximum stable value, the increase may not be as large.

Nevertheless, this is a serious problem when one cannot tolerate a large emittance growth. In order to find a more realistic growth factor for a beam (and not for a particle), we take 5,000 particles distributed uniformly in (ψ_x, ψ_y) but exponentially in E_x and E_y :

$$\rho(E_x^i, E_y^i) \propto \exp(-E_x^i/2\sigma_x^2) \exp(-E_y^i/2\sigma_y^2)$$

where $0 < E_x < 6\sigma_x^2$ and $0 < E_y < 6\sigma_y^2$. The growth factor for the beam as a whole in the horizontal direction is defined as

$$(\sum_k E_{x,k}^{max}) / (\sum_k E_{x,k}^i); \quad k = 1, 5000$$

where $E_{x,k}^i$ and $E_{x,k}^{max}$ are the initial and the maximum possible emittances, respectively, of the k-th particle. If the resonance width is redefined such that the beam emittance growth is no more than 10% in both directions, this width (call it $\epsilon_{10\%}$) is in general larger than ϵ_S and of course larger than ϵ_G . Actually, $\epsilon_{10\%}$ is larger than ϵ_S only for a few low-order resonances:

n_1	3	1	4	1
n_2	0	2	0	3
$\epsilon_{10\%}/\epsilon_S$	2.56	2.33	1.29	1.01

Again it is important to realize that any analytical estimates made in this report are based on the two approximations and one must resort to numerical simulations by a computer in order to find a more reliable emittance growth.

¹ Width of a difference resonance is a rather nebulous quantity. Guignard seems to feel differently; he defines the width but the precise physical meaning of his width is not clear.

² S. Ohnuma, IEEE Trans. Nucl. Sci., NS-28 (1981), p. 2491; Fermilab report TM-988.

³ When the closed orbit is not centered (field error or $\Delta p/p \neq 0$), one must redefine all multipole fields including the quadrupole component around the off-centered closed orbit. All linear lattice parameters are then modified. Since we do not allow any driving terms for linear coupling resonances in the Hamiltonian (the approximation 1), there are no ambiguities in the definition of linear parameters.